

# Kähler Geometry, Momentum Maps and Convex Sets

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October 13, 2015

## Abstract

These notes grew out of an exposé on M. Gromov’s paper “Convex sets and Kähler manifolds” (“Advances in Differential Geometry and Topology,” World Scientific, 1990) at the DMV-Seminar on “Combinatorial Convex Geometry and Toric Varieties” in Blaubeuren in April ‘93. Gromov’s paper deals with a proof of Alexandrov–Fenchel type inequalities and the Brunn–Minkowski inequality for finite dimensional compact convex sets and their variants for compact Kähler manifolds. The emphasis of these notes lies on basic details and the techniques from various mathematical areas involved in Gromov’s arguments.

## Introduction

These notes grew out of an exposé of the author on M. Gromov’s paper [Gr90] at the DMV-Seminar on “Combinatorial Convex Geometry and Toric Varieties” in Blaubeuren in April ‘93. Gromov’s paper deals with connections of *Alexandrov–Fenchel type inequalities* and the *Brunn–Minkowski inequality* (for subsets of  $\mathbb{R}^n$ ) with inequalities for the volume of irreducible complex projective varieties obtained by B. Tesser [Te82] and A. Hovanskii [Ho84].

The *Brunn–Minkowski inequality* for two compact convex subsets  $Y_1, Y_2 \subseteq \mathbb{R}^n$  asserts that their Minkowski sum  $Y_1 + Y_2 := \{a + b : a \in Y_1, b \in Y_2\}$  satisfies

$$\text{vol}(Y_1 + Y_2)^{\frac{1}{n}} \geq \text{vol}(Y_1)^{\frac{1}{n}} + \text{vol}(Y_2)^{\frac{1}{n}}. \quad (1)$$

For bounded convex subsets  $Y_1, \dots, Y_k \subseteq \mathbb{R}^n$  and a multiindex  $J = (j_1, \dots, j_k) \in \mathbb{N}_0^k$ , the *mixed volumes*  $[Y^J] := [Y_1^{j_1}, \dots, Y_k^{j_k}]$  are defined by the coefficients of the homogeneous polynomial

$$\text{vol}(t_1 Y_1 + \dots + t_k Y_k) = \sum_{|J|=n} b_J t^J [Y_1^{j_1}, \dots, Y_k^{j_k}], \quad b_J := \frac{n!}{j_1! \cdots j_k!},$$

on  $(\mathbb{R}_+)^n$ . The *Alexandrov–Fenchel Theorem for convex sets* asserts that, for  $J \in \Delta_k^n$  (the set of all multiindices of degree  $|J| \leq n$  in  $\mathbb{N}_0^k$ ) the corresponding mixed volumes  $[Y^J]$  define a function  $\Delta_k^n \rightarrow \mathbb{R}, J \mapsto \log[Y^J]$  which is concave on every “discrete line” parallel to the edges.

In [Gr90] these inequalities are transferred to compact Kähler manifolds, where one obtains the following *Brunn–Minkowski inequality for Kähler manifolds* (Theorem 4.13): Let

$W_1, \dots, W_k$  be compact  $n$ -dimensional Kähler manifolds and  $M \subseteq W_1 \times \dots \times W_k$  be a compact connected complex submanifold of complex dimension  $n$  and  $M_j \subseteq W_j$  the projection of  $M$  to  $W_j$ . Then

$$\text{vol}(M)^{\frac{1}{n}} \geq \sum_{j=1}^k \text{vol}(M_j)^{\frac{1}{n}}. \quad (2)$$

Here one has equality for  $n = 1$ . Replacing the Kähler manifolds  $W_j$  by projective spaces and  $M$  by an irreducible projective variety, the inequality remains valid. For  $n = 2$  this is the Hodge index theorem [GH78], and for  $n \geq 3$  and  $k = 2$ , this inequality is due to Hovanskii and Tesser [Te82, Ho84].

The central point of [Gr90] is to establish connections between the following three contexts:

- (A) Convex sets,
- (B) Convex functions, and
- (C) Kähler structures

These connections are so tight that they permit us to translate certain theorems such as the Brunn–Minkowski inequality which originally belongs to (A) to the areas (B) and (C). The passage from (A) to (B) is based on Legendre and Laplace transforms and representing convex sets as the image of the differential of a convex function (Fenchel’s Convexity Theorem 1.12). The passage from (B) to (C) is based on Kähler potentials on tube domains  $T_{\mathcal{D}} = \mathcal{D} + i\mathbb{R}^n$  and the quotients  $T_{\mathcal{D}}/i\mathbb{Z}^n$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is an open convex subset, and the corresponding Hamiltonian actions of  $\mathbb{R}^n$ , respectively the torus  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ , obtained from translations in the imaginary direction. Finally, the passage from (C) back to (A) is established by momentum maps for Hamiltonian  $\mathbb{T}^n$ -actions on toric varieties (which correspond to the case  $\mathcal{D} = \mathbb{R}^n$ ) and their convexity properties as developed in [At82, At83, GS82] for compact manifolds and for non-compact manifolds and proper momentum maps in [HNP94]. For a generalization from abelian to real reductive groups and corresponding gradient maps, see [HS10].

In the first three sections we describe these three translation mechanisms. In Section 4 we turn to Gromov’s results and explain how the Brunn–Minkowski inequality is transferred to (B) and (C). Eventually we explain how it fits into a broader context also including a version of the Alexandrov–Fenchel inequality in (C). The emphasis of this note lies rather on the techniques from various mathematical areas involved than on the final results contained in [Gr90]. This note does not contain new results, but we hope that it supplies useful details in a reasonably self-contained fashion and shed some additional light on how convex geometry interacts with Kähler geometry.<sup>1</sup>

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<sup>1</sup>After we wrote the first version of these notes in 1993, they were a starting point for Chapter V on convex sets and functions in the monograph [Ne00]. So we added references to this book in Section 2. We also added a last section on related subjects and more recent developments in this area. The work in [Ne98, Ne99] was also very much inspired by [Gr90]. It constitutes an extension of the connections between (A), (B) and (C) to the context of non-compact non-abelian Lie groups.

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## 1 Convex sets and convex functions

In this section we establish a translation mechanism between convex sets and convex functions.

**Definition 1.1.** Let  $V$  be a real finite dimensional vector space. A function  $f: V \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$  is called *convex* if its *epigraph*

$$\text{epi}(f) := \{(v, t) \in V \times \mathbb{R} : f(v) \leq t\}$$

is a convex set and the set  $D_f = f^{-1}(\mathbb{R})$ , the *domain* of  $f$ , is non-empty. We say that  $f$  is *closed* if  $\text{epi}(f)$  is closed.

If  $C$  is a convex subset of  $V$  and  $f$  is a convex function on  $C$ , then we think of  $f$  as a function  $f: V \rightarrow \mathbb{R}_\infty$  by extending  $f$  to take the value  $\infty$  on the complement of  $C$ . Note that this does not enlarge the epigraph.

**Remark 1.2.** If the convex function  $f$  is not closed, then we can always consider the closure  $\overline{\text{epi}(f)}$  of its epigraph, which is a closed convex subset of  $V \times \mathbb{R}$ . One can show that this set is the epigraph of the convex function

$$\overline{f}(x) := \inf\{t : (x, t) \in \overline{\text{epi}(f)}\}$$

([Ne00, Prop. V.3.7]).

Let  $U \subseteq V$  be an open convex set. Further we consider a function  $f \in C^2(U)$  which has the property that for each  $x \in U$  the symmetric bilinear form  $\mathbf{d}^2 f(x)$  is positive definite. This implies in particular that  $f$  is a strictly convex function in  $U$  because for each pair of different points  $x, y \in U$  the function  $t \mapsto f(tx + (1-t)y)$  on  $]0, 1[$  has a positive second derivative.

**Lemma 1.3.** *The differential  $\mathbf{d}f: U \rightarrow V^*, x \mapsto \mathbf{d}f(x)$  maps  $U$  diffeomorphically onto the open subset  $\mathbf{d}f(U)$  of  $V^*$ .*

*Proof.* It is clear that  $\mathbf{d}f$  is a  $C^1$ -map. Since the second differential  $\mathbf{d}^2 f(x)$  is regular for each  $x \in U$ , it follows from the inverse function theorem that  $\mathbf{d}f$  has a local inverse everywhere. We conclude in particular that  $\mathbf{d}f(U)$  is an open set.

To see that  $\mathbf{d}f$  maps  $U$  diffeomorphically onto its image, it remains to prove injectivity. This follows from

$$\langle \mathbf{d}f(y) - \mathbf{d}f(x), y - x \rangle = \int_0^1 \mathbf{d}^2 f(x + t(y - x))(y - x, y - x) dt > 0$$

for  $x \neq y$ . □

We are interested in properties of open subset  $\mathbf{d}f(U)$  of  $V^*$  and in particular in those cases where this set is convex. This is not always the case (cf. [Ef78]), so we will have to impose a stronger condition on the function  $f$ . To see what this condition could be, we first have to recall some concepts from Fenchel's duality theory for convex functions.

**Definition 1.4.** Let  $f$  be a convex function on  $V$ . Then

$$f^*: V^* \rightarrow \mathbb{R}_\infty, \quad f^*(\alpha) := \sup(\alpha - f)$$

is called the *conjugate* of  $f$ .

The following proposition shows that the passage from  $f$  to  $f^*$  is similar to the passage to the adjoint of an unbounded linear operator on a Hilbert space (cf. [Ru73]).

**Proposition 1.5.** ([Ro70, Th. 12.2]) *The conjugate  $f^*$  of the convex function  $f$  on  $V$  has the following properties:*

- (i)  $f^*$  is a closed convex function.
- (ii)  $f^*(\alpha) + f(x) \geq \alpha(x)$  for all  $x \in V, \alpha \in V^*$ .
- (iii)  $(f^*)^* = \overline{f}$ .

*Proof.* (i) A pair  $(\alpha, t)$  is contained in  $\text{epi}(f^*)$  if and only if  $f^*(\alpha) = \sup(\alpha - f) \leq t$ , which is equivalent to

$$(\forall x \in D_f) \quad t - \alpha(x) \geq -f(x). \tag{3}$$

Therefore  $\text{epi}(f^*)$  is an intersection of a family of closed half spaces and therefore a closed convex set.

(ii) is immediate from the definition of  $f^*$ .

(iii) In view of (3),  $(\alpha, t) \in \text{epi}(f^*)$  if and only if  $f \geq \alpha - t$  on  $V$ , and this is equivalent to  $\overline{f} \geq \alpha - t$ , so that  $f^* = (\overline{f})^*$ .

Next we observe that  $\overline{f} \geq \alpha - f^*(\alpha)$  for all  $\alpha \in V^*$  leads to  $\overline{f} \geq f^{**}$ . If  $\alpha + t \leq \overline{f}$ , then  $f^*(\alpha) \leq -t$ , so that  $\alpha(x) \leq f^*(\alpha) + f^{**}(x) \leq f^{**}(x) - t$ , i.e.,  $\alpha + t \leq f^{**}$ . Now the assertion follows from

$$\overline{f} = \sup\{\alpha + t: \alpha + t \leq f\} = \sup\{\alpha + t: \alpha + t \leq \overline{f}\} = \sup\{\alpha + t: \alpha + t \leq f^{**}\} = f^{**}$$

([Ne00, Lemma V.3.9(ii)]). □

**Definition 1.6.** Let  $f: V \supseteq D_f \rightarrow \mathbb{R}$  be a convex function on  $V$ . Then every linear functional  $\alpha \in V^*$  with

$$f(y) \geq f(x) + \alpha(y - x) \quad \forall y \in D_f,$$

is called a *subgradient of  $f$  at  $x$* . Note that the preceding condition is equivalent to

$$f^*(\alpha) = \max(\alpha - f) = \alpha(x) - f(x).$$

The set of all subgradients of  $f$  at  $x$  is called the *subdifferential of  $f$  at  $x$* . It is denoted  $\partial f(x)$ . It is easy to see that  $\partial f(x) = \{\mathbf{d}f(x)\}$  if  $f$  is differentiable in  $x$  (cf. [Ro70, Th. 25.1]). Note that  $\partial f(x) = \emptyset$  if  $x \notin D_f$ .

The following proposition makes the symmetry between  $f$  and  $f^*$  apparent.

**Proposition 1.7.** ([Ro70, Th. 23.5], [Ne00, Prop. V.3.22]) *Let  $f$  be a closed convex function on  $V$ . Then the following are equivalent:*

- (1)  $\alpha \in \partial f(x)$ .
- (2)  $f(x) + f^*(\alpha) = \alpha(x)$ .
- (3)  $x \in \partial f^*(\alpha)$ .

**Definition 1.8.** (a) In the following we write  $\text{algint}(C)$  for the relative interior (also called the *algebraic interior*) of a convex set  $C$  with respect to the affine subspace it generates.

(b) A subset  $C$  of a vector space  $V$  is said to be *almost convex* if  $\overline{C}$  is convex and  $\text{algint}\overline{C}$  is contained in  $C$ .

The following observation basically asserts the convexity of the range of the multi-valued mapping  $\partial f$ . It is a first approach to the result on the convexity of the image of  $\mathbf{d}f$  for differentiable convex functions.

**Corollary 1.9.** *If  $f$  is a closed convex function, then  $\text{im}(\partial f)$  is an almost convex dense subset of  $D_{f^*}$ .*

*Proof.* ([Ne00, Cor. V.3.23]) In view of Proposition 1.7,  $\alpha \in \text{im}(\partial f)$  if and only if  $\partial f^*(\alpha) \neq \emptyset$ . On the other hand,  $\partial f^*(\alpha) = \emptyset$  for  $\alpha \notin D_{f^*}$  and  $\partial f^*(\alpha) \neq \emptyset$  for  $\alpha \in \text{algint}D_{f^*}$  ([Ro70, Th. 23.4] or [Ne00, Lemma V.3.21(iii)]). Therefore  $\text{im}(\partial f)$  is an almost convex set contained in  $D_{f^*}$  which contains  $\text{algint}(D_{f^*})$ .  $\square$

This corollary shows us that in the case where  $f$  is differentiable on the interior of its domain, the difficulties are caused by the sets  $\partial f(x)$  for  $x \in \partial U$ . So we need a condition which guarantees that these sets do not contribute to the image of  $\partial f$ .

**Definition 1.10.** Let  $U \subseteq V$  be an open convex subset. A function  $f \in C^2(U)$  with  $\mathbf{d}^2 f(x)$  positive definite for all  $x \in U$  is called a  *$C^2$ -Legendre function* if for every  $x \in U$  and every  $y \in \partial U$  we have that

$$\lim_{t \rightarrow 1^-} \mathbf{d}f(x + t(y - x))(y - x) = \infty.$$

We will see that the  $C^2$ -Legendre functions are those which are of interest for our purposes.

**Lemma 1.11.** ([Ro70, Th. 26.1] or [Ne00, Lemma V.3.30]) *If  $f$  is a closed convex function such that  $\text{int}(D_f) \neq \emptyset$  and the restriction of  $f$  to  $\text{int}(D_f)$  is a  $C^2$ -Legendre function, then  $\partial f(x) = \emptyset$  for  $x \in \partial D_f$ .*

The following theorem is a sharper version of a result of Fenchel (cf. [Fe49]). Fenchel did assume that  $f$  tends to infinity at the boundary of  $U$  which is not necessary.

**Theorem 1.12.** (Fenchel's Convexity Theorem <sup>2</sup>) *Let  $U$  be an open convex set and  $f: U \rightarrow \mathbb{R}$  a  $C^2$ -Legendre function. Then  $\mathbf{d}f(U) = \text{int}(D_{f^*})$  is an open convex set,  $\mathbf{d}f$  maps  $U$  diffeomorphically onto  $\mathbf{d}f(U)$ ,  $f^*|_{\text{int}(D_{f^*})}$  is  $C^2$ , and  $\mathbf{d}(f^*): \mathbf{d}f(U) \rightarrow U$  is an inverse of  $\mathbf{d}f$ .*

*Proof.* We know from Lemma 1.3 that  $\mathbf{d}f(U)$  is an open set. Further,  $\bar{f}$  is a closed convex function,  $\text{epi}(\bar{f}) = \overline{\text{epi}(f)}$ ,  $U$  is dense in  $D_{\bar{f}}$ , and  $\bar{f}|_U = f$  ([Ne00, Prop. V.3.2(iii)]).

Now we use Lemma 1.11, to see that  $\text{im}(\partial \bar{f}) = \mathbf{d}f(U)$ , so that  $\mathbf{d}f(U) = \text{int}(D_{f^*})$  by Corollary 1.9 since an open dense almost convex subset of  $D_{f^*}$  must be equal to  $\text{int}(D_{f^*})$ .

In view of Proposition 1.7, it is clear that  $\partial f^*(\alpha) = \emptyset$  for  $\alpha \notin \text{int}(D_{f^*})$ , that  $\partial f^*$  is single valued on  $\text{int} D_{f^*}$ , and that it is an inverse of the function  $\mathbf{d}f: U \rightarrow \text{int}(D_{f^*})$ .  $\square$

**Corollary 1.13.** *Let  $f \in C^2(V)$  be such that  $\mathbf{d}^2 f(x)$  is positive definite for all  $x \in V$ . Then  $\mathbf{d}f(V) = \text{int} D_{f^*}$  is an open convex set,  $\mathbf{d}f$  maps  $V$  diffeomorphically onto  $\mathbf{d}f(V)$ , and  $\mathbf{d}(f^*): \mathbf{d}f(V) \rightarrow V$  is an inverse of  $\mathbf{d}f$ .*

*Proof.* This is an immediate consequence of Theorem 1.12 because in the case where  $U = V$  the additional condition for a Legendre function is trivially satisfied.  $\square$

So far we have considered general convex functions. Now we turn to the special class of Laplace transforms of (positive) measures.

## Laplace transforms

Let  $V$  be a finite dimensional real vector space and  $\mu$  a non-zero positive Borel measure on  $V^*$ . We define the *Laplace transform* of  $\mu$  to be the function

$$\mathcal{L}(\mu): V \rightarrow ]0, \infty], \quad x \mapsto \int_{V^*} e^{\alpha(x)} d\mu(\alpha).$$

**Definition 1.14.** We define the functions  $e_x(\alpha) := e^{\alpha(x)}$  on  $V^*$ . Note that  $\mathcal{L}(\mu)(x) > 0$  for all  $x \in V$  since  $\mu \neq 0$ . We say that  $\mu$  is *admissible* if there exists an  $x \in V$  such that  $\mathcal{L}(\mu)(x) < \infty$ . If  $\mu$  is admissible, then there exists an  $x \in V$  such that  $e_x \mu$  is a finite measure and since  $e_x$  is bounded from below on every compact set, it follows in particular that  $\mu$  is a Radon measure and therefore  $\sigma$ -finite. We write  $C_\mu$  for the closed convex hull of the support of  $\mu$ . This is the smallest closed convex subset of  $V^*$  such that its complement is a  $\mu$ -null set.

**Proposition 1.15.** (i) *The functions  $\mathcal{L}(\mu)$  and  $\log \mathcal{L}(\mu)$  are closed, convex and if  $C_\mu$  has interior points, then  $\mathcal{L}(\mu)$  is strictly convex on  $D_{\mathcal{L}(\mu)}$ .*

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<sup>2</sup>See [Gr90] for a  $C^1$ -version of this theorem.

- (ii) The function  $\mathcal{L}(\mu)$  is analytic on  $\text{int } D_{\mathcal{L}(\mu)}$  and it has a holomorphic extension to the tube domain  $\text{int } D_{\mathcal{L}(\mu)} + iV$ .
- (iii) If  $C_\mu$  has interior points, then the bilinear form  $\mathbf{d}^2(\log \mathcal{L}(\mu))(x)$  is positive definite for all  $x \in \text{int } D_{\mathcal{L}(\mu)}$ .
- (iv)  $D_{\mathcal{L}(\mu)}^*$  is a convex set which is dense in  $C_\mu$ .

*Proof.* (i) [BN78, Th. 7.1] or [Ne00, Prop. V.4.3, Cor. V.4.4]

(ii) [BN78, Th. 7.2] or [Ne00, Cor. V.4.4, Prop. V.4.6]

(iii) ([Ne00, Prop. V.4.6]) In view of (ii), the function  $\log \mathcal{L}(\mu)$  is analytic on  $\text{int } D_\mu$ . We calculate

$$\mathbf{d}(\log \mathcal{L}(\mu))(x) = \frac{1}{\mathcal{L}(\mu)(x)} \mathbf{d}\mathcal{L}(\mu)(x) = \frac{1}{\mathcal{L}(\mu)(x)} \int_{C_\mu} \alpha e^{\alpha(x)} d\mu(\alpha).$$

Hence

$$\begin{aligned} & \mathbf{d}^2(\log \mathcal{L}(\mu))(x)(y, y) \\ &= -\frac{1}{\mathcal{L}(\mu)(x)^2} \left( \int_{C_\mu} \alpha(y) e^{\alpha(x)} d\mu(\alpha) \right)^2 + \frac{1}{\mathcal{L}(\mu)(x)} \int_{C_\mu} \alpha(y)^2 e^{\alpha(x)} d\mu(\alpha). \end{aligned}$$

To see that this expression is positive for  $0 \neq y$ , let  $g(\alpha) := e^{\frac{1}{2}\alpha(x)}$  and  $h(\alpha) = \alpha(y)g(\alpha)$ . Then

$$\begin{aligned} & \mathcal{L}(\mu)(x)^2 \mathbf{d}^2(\log \mathcal{L}(\mu))(x)(y, y) \\ &= - \left( \int_{C_\mu} g(\alpha) h(\alpha) d\mu(\alpha) \right)^2 + \int_{C_\mu} g(\alpha)^2 d\mu(\alpha) \int_{C_\mu} h(\alpha)^2 d\mu(\alpha) \geq 0 \end{aligned}$$

by the Cauchy–Schwartz inequality. In the case of equality it follows that

$$\alpha \mapsto h(\alpha)g(\alpha)^{-1} = \alpha(y)$$

is a  $\mu$ -almost everywhere constant function, i.e.,  $\text{supp}(\mu)$  lies in an affine hyperplane, contradicting the hypothesis that  $C_\mu$  has interior points.

(iv) [BN78, Th. 9.1] □

Note that Proposition 1.15(iv) shows in particular that the image of  $\mathbf{d}\mathcal{L}(\mu)$  is contained in the convex set  $C_\mu$ . The following results gives a good criterion for the convexity of the set  $\mathbf{d}\mathcal{L}(\mu)(\text{int } D_{\mathcal{L}(\mu)})$ :

**Theorem 1.16.** (Convexity Theorem for Laplace transforms) *Suppose that  $\text{int } D_{\mathcal{L}(\mu)} \neq \emptyset$  and  $\text{int } C_\mu \neq \emptyset$ . Then*

$$\mathbf{d}\mathcal{L}(\mu)(\text{int } D_{\mathcal{L}(\mu)}) = \text{int } C_\mu$$

*if and only if the restriction of  $\mathcal{L}(\mu)$  to  $\text{int } D_\mu$  is a Legendre function.*

*If only  $\text{int } D_{\mathcal{L}(\mu)} \neq \emptyset$  and  $\alpha_0 + H$  is the affine subspace generated by  $C_\mu$ , then  $\mathcal{L}(\mu)$  factors to a function on  $V/H^\perp$  and the first assertion applies to the translated measure  $(-\alpha_0)^*\mu$  on  $H \cong (V/H^\perp)^*$ .*

*Proof.* ([Ne00, Thm. V.4.9]) The first part follows from [BN78, Th. 9.2]. For the second part we note that for  $y \in H^\perp$  we clearly have that

$$\mathcal{L}(\mu)(x+y) = \int_{C_\mu} e^{\alpha(x+y)} d\mu(\alpha) = e^{\alpha_0(y)} \int_{C_\mu} e^{\alpha(x)} d\mu(\alpha) = e^{\alpha_0(y)} \mathcal{L}(\mu)(\alpha).$$

Translating  $\mu$  by  $-\alpha_0$ , we obtain a measure  $\mu'$  which is supported by the subspace  $H$ ,  $\mathcal{L}(\mu')(x) = e^{-\alpha_0(x)} \mathcal{L}(\mu)(x)$  is its Laplace transform which in fact factors to a function on  $V/H^\perp$ , where it is the Laplace transform of  $\mu'$ , and  $\mathbf{d}(\log \mathcal{L}(\mu'))(x) = -\alpha_0 + \mathbf{d}(\log \mathcal{L}(\mu))(x)$ . This proves the second part since now the first part applies to the measure  $\mu'$ .  $\square$

**Corollary 1.17.** *If  $D_{\mathcal{L}(\mu)} = V$ , then  $\mathbf{d}\mathcal{L}(\mu)(V) = \text{algint}(C_\mu)$ .*

**Theorem 1.18.** *For every open convex subset  $C \subseteq V^*$  there exists a  $C^2$ -Legendre function  $f$  on  $V$  such that  $\mathbf{d}f(V) = C$ .*

*Proof.* ([Ne00, Prop. V.4.14]) If  $C$  is bounded, then we define  $\mu$  to be Lebesgue measure on  $C$ . In general we identify  $V$  and  $V^*$  with  $\mathbb{R}^n$ , and define  $\mu$  by  $d\mu(y) = e^{-\|y\|^2} dy$  on  $C$  and zero outside of  $C$ . Then  $D_{\mathcal{L}(\mu)} = V$  and  $\text{int } C_\mu = C \neq \emptyset$ . Therefore  $\mathbf{d}\mathcal{L}(\mu)(V) = C$  by Corollary 1.17.  $\square$

For later applications in Section 4 we record the following two lemmas.

**Lemma 1.19.** *If  $f_1$  and  $f_2$  are  $C^2$ -Legendre functions on  $V$ , then*

$$\mathbf{d}(f_1 + f_2)(V) = \mathbf{d}f_1(V) + \mathbf{d}f_2(V).$$

*Proof.* The inclusions

$$\mathbf{d}(f_1 + f_2)(V) \subseteq \mathbf{d}f_1(V) + \mathbf{d}f_2(V) \subseteq D_{f_1^*} + D_{f_2^*} \subseteq D_{(f_1+f_2)^*}$$

follow directly from the definitions and Proposition 1.7. Since  $f_1 + f_2$  is also a  $C^2$ -Legendre function on  $V$ , we conclude that  $\mathbf{d}(f_1 + f_2)(V) = \text{int } D_{(f_1+f_2)^*}$  so that the left most set is dense in the right most set. Now the fact that  $\mathbf{d}f_1(V) + \mathbf{d}f_2(V)$  is an open convex set proves the equality with  $\mathbf{d}(f_1 + f_2)(V)$ .  $\square$

**Lemma 1.20.** *Let  $U \subseteq \mathbb{R}^n$  be an open convex set and  $f \in C^2(U)$  such that  $\mathbf{d}^2 f(x)$  is positive definite for all  $x \in U$ . Then*

$$\text{vol}(\mathbf{d}f(U)) = \int_U \det(\mathbf{d}^2 f)(x) dx,$$

where we identify  $\mathbf{d}^2 f$  with the matrix whose entries are  $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}$ .

*Proof.* Since  $\mathbf{d}^2 f(x)$  is the Jacobian of the mapping  $\mathbf{d}f: U \rightarrow \mathbb{R}^n$ , and it has positive determinant, the assertion follows from the transformation theorem for integrals:

$$\text{vol}(\mathbf{d}f(U)) = \int_{\mathbf{d}f(U)} 1 \, dx = \int_U |\det(\mathbf{d}^2 f(x))| \, dx = \int_U \det(\mathbf{d}^2 f(x)) \, dx. \quad \square$$



## 2 Convex functions and Kähler structures

Now we turn to the relations between convex functions and Kähler structures. A good reference for foundational material concerning this section is Chapter 0.2 in [GH78].

We consider  $\mathbb{C}^n \cong \mathbb{R}^{2n}$  as a  $2n$ -dimensional real vector space and write  $J$  for the linear mapping representing the multiplication by  $i$ .

**Definition 2.1.** (a) A skew symmetric real bilinear form  $\omega$  on  $V = \mathbb{R}^{2n}$  is said to be

- (1) *positive* if  $\omega(v, Jv) \geq 0$  holds for all  $v \in V$ .
- (2) *strictly positive* if  $\omega(v, Jv) > 0$  holds for all  $v \in V \setminus \{0\}$ .
- (3) a  $(1, 1)$ -form if  $\omega(Jv, Jw) = \omega(v, w)$  holds for all  $v, w \in V$ .

Note that  $\omega$  is a  $(1, 1)$ -form if and only if  $h(v, w) := \omega(v, Jw)$  defines a real symmetric bilinear form on  $V$ .

(b) Using the concepts of (a) in each point, we get similar concepts for differential 2-forms on open subsets of  $\mathbb{C}^n$  and more generally on a complex manifold (one only needs an almost complex structure).

Let  $M$  be a complex manifold and  $J$  the corresponding almost complex structure, i.e., for every  $x \in M$  the mapping  $J_x: T_x(M) \rightarrow T_x(M)$  represents multiplication by  $i$ . Then  $J$  acts simply by multiplication on vectors, hence on tensors, and via duality its also acts on differential forms such that the pairing between forms and vectors is invariant under  $J$ . It follows in particular that, if  $\omega$  is a 1-form and  $\mathcal{X}$  a vector field,

$$\langle J\omega, \mathcal{X} \rangle = \langle \omega, J^{-1}\mathcal{X} \rangle = -\langle \omega, J\mathcal{X} \rangle.$$

Let  $\mathcal{E}^{(p,q)}(M)$  denote the space of  $(p, q)$ -forms on  $M$ . Recall that a  $(p, q)$ -form can be expressed in local coordinates  $z_1, \dots, z_n$  as a sum

$$\omega = \sum_{I, J} f_{I, J} dz_I \wedge d\bar{z}_J,$$

where  $I = (i_1, \dots, i_p) \in \{1, \dots, n\}^p$  and  $J = (j_1, \dots, j_q) \in \{1, \dots, n\}^q$  are multiindices,  $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ ,  $d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ , and the  $f_{I, J}$  are smooth functions.

Since

$$J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \quad \text{and} \quad J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j},$$

it follows that  $Jdx_j = dy_j$  and  $Jdy_j = -dx_j$  which in turn yields

$$Jdz_j = -idz_j \quad \text{and} \quad Jd\bar{z}_j = id\bar{z}_j.$$

Thus  $J\omega = i^{q-p}\omega$  holds for every  $(p, q)$ -form  $\omega$  and the  $J$ -invariant forms are precisely those with  $p \equiv q \pmod{4}$ .

Next we decompose the exterior derivative  $d: \mathcal{E}^m \rightarrow \mathcal{E}^{m+1}$  as  $d = \partial + \bar{\partial}$ , where  $\partial\mathcal{E}^{(p,q)} \subseteq \mathcal{E}^{(p+1,q)}$  and  $\bar{\partial}\mathcal{E}^{(p,q)} \subseteq \mathcal{E}^{(p,q+1)}$ . On functions we have locally

$$\partial f = \sum_j \frac{\partial f}{\partial z_j} dz_j, \quad \bar{\partial} f = \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \quad (4)$$

and on a  $(p, q)$ -form  $\omega = \sum_{I, J} f_{I, J} \mathbf{d}z_I \wedge \mathbf{d}\bar{z}_J$  we find

$$\partial\omega = \sum_{I, J} \partial f_{I, J} \wedge \mathbf{d}z_I \wedge \mathbf{d}\bar{z}_J \quad \text{and} \quad \bar{\partial}\omega = \sum_{I, J} \bar{\partial} f_{I, J} \wedge \mathbf{d}z_I \wedge \mathbf{d}\bar{z}_J.$$

Note that  $\mathbf{d}^2 = 0$  yields  $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$ .

**Lemma 2.2.** *Let  $f$  be a smooth function on the complex manifold  $M$ . Then the following assertions hold:*

- (i)  $\mathbf{d}J\mathbf{d}f = 2i\partial\bar{\partial}f$  is a  $(1, 1)$ -form and in particular  $J$ -invariant.
- (ii) If  $h_f$  is the symmetric form defined by

$$h_f(v, w) := \mathbf{d}J\mathbf{d}f(v, Jw),$$

then  $h_f = \mathbf{d}^2 f + J\mathbf{d}^2 f$  in local coordinates.

*Proof.* (i) We write  $\mathbf{d}f = \partial f + \bar{\partial}f$ . Then

$$J\mathbf{d}f = J\partial f + J\bar{\partial}f = -i\partial f + i\bar{\partial}f = i(\bar{\partial} - \partial)f.$$

Therefore

$$\mathbf{d}J\mathbf{d}f = i(\partial + \bar{\partial})(\bar{\partial} - \partial)f = 2i\partial\bar{\partial}f.$$

Since  $\bar{\partial}f$  is a  $(0, 1)$ -form,  $\partial\bar{\partial}f$  is a  $(1, 1)$ -form and the assertion follows.

(ii) First we calculate

$$\mathbf{d}J\mathbf{d}f = 2i \sum_{j, k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \mathbf{d}z_j \wedge \mathbf{d}\bar{z}_k.$$

Writing

$$\mathbf{d}z_j \wedge \mathbf{d}\bar{z}_k = \mathbf{d}z_j \otimes \mathbf{d}\bar{z}_k - \mathbf{d}\bar{z}_k \otimes \mathbf{d}z_j,$$

we find with (4) that

$$\begin{aligned} h_f &= 2i \sum_{j, k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} (\mathbf{d}z_j \otimes J^\top \mathbf{d}\bar{z}_k - \mathbf{d}\bar{z}_k \otimes J^\top \mathbf{d}z_j) \\ &= 2i \sum_{j, k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} (-i\mathbf{d}z_j \otimes \mathbf{d}\bar{z}_k - i\mathbf{d}\bar{z}_k \otimes \mathbf{d}z_j) \\ &= 2 \sum_{j, k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} (\mathbf{d}z_j \otimes \mathbf{d}\bar{z}_k + \mathbf{d}\bar{z}_k \otimes \mathbf{d}z_j). \end{aligned}$$

On the other hand we have

$$\mathbf{d}^2 f = \sum_{j, k} \left( \frac{\partial^2 f}{\partial z_j \partial z_k} \mathbf{d}z_j \otimes \mathbf{d}z_k + \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} (\mathbf{d}z_j \otimes \mathbf{d}\bar{z}_k + \mathbf{d}\bar{z}_k \otimes \mathbf{d}z_j) + \frac{\partial^2 f}{\partial \bar{z}_j \partial \bar{z}_k} \mathbf{d}\bar{z}_j \otimes \mathbf{d}\bar{z}_k \right).$$

In  $\mathbf{d}^2 f + J\mathbf{d}^2 f$ , the first and the last term cancel and we obtain  $\mathbf{d}^2 f + J\mathbf{d}^2 f = h_f$ . □

Note that the definition of  $H$  rests on the choice of local coordinates but that the definition of  $h_f$  is coordinate free.

We pave the way to later applications with the following proposition which relates Legendre functions to exact positive  $(1, 1)$ -forms on complex tori.

**Proposition 2.3.** *Let  $M = \mathbb{C}^n / i\mathbb{Z}^n \cong (\mathbb{C}^\times)^n$  be endowed with the usual complex structure.*

- (i) *Let  $U \subseteq \mathbb{R}^n$  be an open subset and  $f \in C^\infty(U)$  be such that  $\mathbf{d}^2 f(x)$  is positive semidefinite for all  $x \in U$ . We define the function  $\tilde{f}$  on  $M_U := \{z + i\mathbb{Z}^n : \operatorname{Re} z \in U\}$  by  $\tilde{f}(z + i\mathbb{Z}^n) := f(\operatorname{Re} z)$ . Then  $\omega_f := \mathbf{d}J\mathbf{d}\tilde{f}$  is a positive  $(1, 1)$ -form which is strictly positive if and only if  $\mathbf{d}^2 f$  is everywhere positive definite. Moreover*

$$\frac{1}{n!} \int_{M_U} \omega_f^n = \int_U \det(\mathbf{d}^2 f).$$

- (ii) *If  $\omega$  is a positive exact  $(1, 1)$ -form on  $M$  which is invariant under the action of the torus  $T = i\mathbb{R}^n / i\mathbb{Z}^n$ , then there exists a smooth function  $f$  on  $\mathbb{R}^n$  such that  $\omega = \omega_f$  and  $\mathbf{d}^2 f$  is everywhere positive semidefinite.*

*Proof.* (i) We have already seen in Lemma 2.2 that  $\omega := \omega_f$  is a  $(1, 1)$ -form. Let  $v, w \in \mathbb{R}^n$  and  $m = x + iy + i\mathbb{Z}^n \in M$ . Then we use Lemma 2.2(ii) to see that

$$\begin{aligned} \omega(m)(v + iw, i(v + iw)) &= \omega(m)(v + iw, -w + iv) \\ &= \mathbf{d}^2 \tilde{f}(m)(v + iw, v + iw) + \mathbf{d}^2 \tilde{f}(m)(w - iv, w - iv) \\ &= \mathbf{d}^2 f(x)(v, v) + \mathbf{d}^2 f(x)(w, w) \geq 0. \end{aligned}$$

This calculation also shows that  $\omega$  is strictly positive if and only if  $\mathbf{d}^2 f$  is everywhere positive definite.

To prove the integral formula, we first use the above calculation and polarization to obtain

$$\omega(m)(v + iw, v' + iw') = \mathbf{d}^2 f(x)(v, w') - \mathbf{d}^2 f(x)(w, v').$$

Thus we can write

$$\omega = \sum_{j,k} \frac{\partial^2 f}{\partial x_j \partial x_k} \mathbf{d}x_j \wedge \mathbf{d}y_k. \quad (5)$$

Let  $\alpha_k := \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k} \mathbf{d}x_j$ . Then  $\omega = \sum_k \alpha_k \wedge \mathbf{d}y_k$  and therefore

$$\begin{aligned} \frac{1}{n!} \omega^n &= \alpha_1 \wedge \mathbf{d}y_1 \wedge \alpha_2 \wedge \dots \wedge \mathbf{d}y_n = \sum_{j_1, \dots, j_n=1}^n \frac{\partial^2 f}{\partial x_{j_1} \partial x_1} \mathbf{d}x_{j_1} \wedge \mathbf{d}y_1 \wedge \frac{\partial^2 f}{\partial x_{j_2} \partial x_2} \mathbf{d}x_{j_2} \wedge \dots \wedge \mathbf{d}y_n \\ &= \det(\mathbf{d}^2 f) \mathbf{d}x_1 \wedge \mathbf{d}y_1 \wedge \mathbf{d}x_2 \wedge \dots \wedge \mathbf{d}y_n. \end{aligned}$$

Now we calculate the integral:

$$\frac{1}{n!} \int_M \omega^n = \int_{(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}))^n} \det(\mathbf{d}^2 f)(x) dx_1 dy_1 dx_2 dy_2 \dots dx_n dy_n = \int_{\mathbb{R}^n} \det(\mathbf{d}^2 f)(x) dx.$$

(ii) First we observe that every invariant exact 2-form  $\beta = \mathbf{d}\alpha$  on the torus  $T$  is also the differential of an invariant exact form, hence zero. This can be achieved by averaging  $\alpha$ . On the other hand every invariant 1-form on  $T$  is closed because  $T$  is abelian (cf. [ChE48]). Therefore  $T$  permits no non-zero invariant exact 2-forms.

This shows that the restriction of  $\omega$  to the  $T$ -cosets in  $M$  vanishes. Since  $\omega$  is also  $J$ -invariant, it follows that  $i\mathbb{R}^n$  is also everywhere isotropic for  $\omega$ . Therefore, in coordinates from the mapping  $\mathbb{R}^{2n} \rightarrow M$ , we have that

$$\omega(x) = \sum_{j,k} a_{jk}(x) \mathbf{d}x_j \wedge \mathbf{d}y_k,$$

where we have used the  $T$ -invariance to see that the functions  $a_{jk}$  do not depend on the  $y_k$ 's.

The closedness of  $\omega$ :

$$\mathbf{d}\omega = \sum_{j,k,\ell} \frac{\partial a_{jk}}{\partial x_\ell} \mathbf{d}x_\ell \wedge \mathbf{d}x_j \wedge \mathbf{d}y_k = 0$$

now shows that the 1-forms  $\alpha_k := \sum_{j,k} a_{jk} \mathbf{d}x_j$  are also closed. Hence there exist smooth functions  $f_k$  on  $\mathbb{R}^n$  with  $\mathbf{d}f_k = \alpha_k$ , i.e.,  $a_{jk} = \frac{\partial f_k}{\partial x_j}$ .

The  $J$ -invariance of  $\omega$  further yields

$$\omega = J\omega = \sum_{j,k} a_{jk} \mathbf{d}y_j \wedge (-\mathbf{d}x_k) = \sum_{j,k} a_{jk} \mathbf{d}x_k \wedge \mathbf{d}y_j$$

and therefore  $a_{jk} = a_{kj}$ . Thus

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k}$$

which means that  $\sum_k f_k \mathbf{d}x_k$  is closed. Therefore we find a smooth function  $f$  on  $\mathbb{R}^n$  with  $\mathbf{d}f = \sum_k f_k \mathbf{d}x_k$ . Finally  $\omega = \mathbf{d}J\tilde{f}$  follows from (5).  $\square$

### 3 Kähler structures, Hamiltonian actions and momentum maps

In this section we describe how the momentum map for Hamiltonian group actions can be used to relate Kähler manifolds and convex sets. We apply this in particular to orbits of complex tori in projective spaces. Again we do not intend to describe the geodesic way to the final results but rather to show how the different approaches in the literature fit together.

For more details on symplectic manifolds, momentum maps and Hamiltonian actions we refer to [GS84] and [LM87].

**Definition 3.1.** (a) Let  $M$  be a smooth real manifold. A *symplectic structure* on  $M$  is a closed, non-degenerate 2-form  $\omega$ .

(b) Let  $M$  be a complex manifold. A *Kähler structure* on  $M$  is a strictly positive, closed  $(1,1)$ -form  $\omega$ .

Then  $\omega$  is in particular non-degenerate, so that it defines a symplectic structure on  $M$ . If  $J$  denotes the almost complex structure of  $M$ , then for each  $x \in M$  the sesquilinear form

$$h(x)(v, w) := \omega(x)(v, Jw) - i\omega(x)(v, w)$$

is positive definite, so that it defines a complex Hilbert space structure on the tangent space  $T_x(M)$ .

Let  $(M, \omega)$  be a connected symplectic manifold. Then we can associate to each smooth function  $F \in C^\infty(M)$  a *Hamiltonian vector field*  $\mathcal{X}_F$  which is determined by the equation

$$\mathrm{d}F = -i_{\mathcal{X}_F}\omega.$$

If we define a Lie algebra structure on  $C^\infty(M)$  by the *Poisson bracket*

$$\{F, G\} := \omega(\mathcal{X}_F, \mathcal{X}_G) = \mathcal{X}_F G = -\mathcal{X}_G F,$$

then

$$C^\infty(M) \rightarrow \mathcal{V}(M), \quad F \mapsto \mathcal{X}_F$$

is a homomorphism of Lie algebras whose kernel consists of the constant functions.

Now let  $G$  be a connected Lie group acting on  $(M, \omega)$  and leaving the symplectic structure invariant. Then we obtain a homomorphism of Lie algebras  $\psi: \mathfrak{g} = \mathbf{L}(G) \rightarrow \mathcal{V}(M)$  with

$$\psi(X)_p := \left. \frac{d}{dt} \right|_{t=0} \exp(-tX).p \quad \text{for all } p \in M.$$

Such an action is called *Hamiltonian* if there exists a homomorphism  $\varphi: \mathfrak{g} \rightarrow C^\infty(M)$  in such a way that

$$\mathcal{X}_{\varphi(X)} = \psi(X)$$

holds for all  $X \in \mathfrak{g}$ , i.e.,  $\psi$  lifts to a homomorphism  $\mathfrak{g} \rightarrow (C^\infty(M), \{\cdot, \cdot\})$ .

For a Hamiltonian action one has a mapping

$$\Phi: M \rightarrow \mathfrak{g}^*, \quad \Phi(m)(X) := \varphi(X)(m)$$

called the *momentum map*. It is an equivariant map from  $M$  to  $\mathfrak{g}^*$ , where  $G$  acts on  $\mathfrak{g}^*$  by the coadjoint action  $g.\alpha := \mathrm{Ad}^*(g)\alpha := \alpha \circ \mathrm{Ad}(g)^{-1}$ .

In the last section we have seen how to obtain Kähler structures on a complex manifold  $M$  via smooth functions as  $\omega = \mathrm{d}J\mathrm{d}f$ . Now we bring this together with holomorphic Hamiltonian actions of a connected Lie group  $G$  on  $M$ .

Let  $G \times M \rightarrow M$  be an action of  $G$  on  $M$  by holomorphic mappings and  $\psi: \mathfrak{g} \rightarrow \mathcal{V}(M)$  the corresponding homomorphism into the Lie algebra of vector fields. For  $X \in \mathfrak{g}$  we set

$$\varphi(X) := \langle J\mathrm{d}f, \psi(X) \rangle = -(J\psi(X))f \quad \text{for } f \in C^\infty(M). \quad (6)$$

**Proposition 3.2.** *Suppose that the function  $\psi(X)f$  is constant for every  $X \in \mathfrak{g}$ . Then  $\Phi(m)(X) := \varphi(X)(m)$  defines a momentum map for the action of  $G$  on the symplectic manifold  $(M, \omega)$ .*

*Proof.* Let  $X \in \mathfrak{g}$  and write  $\mathcal{L}_{\psi(X)}$  for the Lie derivative along the vector field  $\psi(X)$ . Then the fact that  $G$  acts holomorphically means that  $\mathcal{L}_{\psi(X)}J = 0$  for all  $X \in \mathfrak{g}$  and therefore

$$\mathcal{L}_{\psi(X)}J\mathrm{d}f = J\mathcal{L}_{\psi(X)}\mathrm{d}f = J\mathrm{d}i_{\psi(X)}\mathrm{d}f = J\mathrm{d}(\psi(X)f) = 0$$

by the Cartan formula. Hence

$$0 = \mathcal{L}_{\psi(X)} J \mathbf{d}f = i_{\psi(X)} \mathbf{d} J \mathbf{d}f + \mathbf{d} i_{\psi(X)} J \mathbf{d}f = i_{\psi(X)} \omega + \mathbf{d}\varphi(X)$$

shows that  $\mathcal{X}_{\varphi(X)} = \psi(X)$ .

It remains to show that  $\varphi$  is a homomorphism of Lie algebras. To this end, we first note that  $J[\psi(X), \psi(Y)] = [J\psi(X), \psi(Y)]$  follows from the fact that  $G$  acts holomorphically because the Lie algebra of all vector fields generating holomorphic flows is a complex Lie subalgebra with respect to the complex structure induced by  $J$ . We also recall that  $\psi(Y)f$  is constant so that  $\mathcal{X}\psi(Y)f = 0$  holds for every vector field  $\mathcal{X}$  on  $M$ . With these remarks in mind we calculate:

$$\begin{aligned} \varphi([X, Y]) &= -(J\psi([X, Y]))f = -(J[\psi(X), \psi(Y)])f = -([J\psi(X), \psi(Y)])f \\ &= \psi(Y)(J\psi(X))f - J\psi(X)(\psi(Y)f) = \psi(Y)(J\psi(X))f = -\psi(Y)\varphi(X) \\ &= -\mathcal{X}_{\varphi(Y)}\varphi(X) = -\{\varphi(Y), \varphi(X)\} = \{\varphi(X), \varphi(Y)\}. \end{aligned} \quad \square$$

**Remark 3.3.** If the vector field  $\psi(X)$  on  $M$  has compact orbits, then the constancy of the function  $\psi(X)f$  implies that even  $\psi(X)f = 0$  because every periodic affine function is constant. It follows in particular that, for a compact group  $G$ , the function  $f$  has to be invariant under the action of  $G$ .

## Projective spaces

A particular interesting case for these constructions is the complex projective space  $\mathbb{P}(\mathbb{C}^n)$ . For a non-zero vector  $v \in \mathbb{C}^n$  we write  $[v] = \mathbb{C}^\times v$  for the corresponding ray in  $\mathbb{P}(\mathbb{C}^n)$  and write  $(v_1 : \dots : v_n)$  for  $[(v_1, \dots, v_n)]$  (homogeneous coordinates).

We want to define a Kähler structure  $\omega$  on the projective space (the *Fubini-Study metric*) (cf. [GH78, p. 30]). The most transparent way to do this is to construct  $\pi^*\omega$  on the unit sphere  $\mathbb{S}^{2n-1}$ , where  $\pi: \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{C}^n)$  is the canonical projection.

We start with the function  $F(z) := \frac{1}{2} \log \|z\|^2$ . Then

$$\mathbf{d}F(z)(v) = \frac{1}{\|z\|^2} \operatorname{Re}\langle z, v \rangle, \quad J\mathbf{d}F(z)(v) = \frac{1}{\|z\|^2} \operatorname{Im}\langle v, z \rangle,$$

and

$$\mathbf{d}J\mathbf{d}F(z)(v, w) = \frac{2}{\|z\|^4} (\operatorname{Re}\langle z, v \rangle \operatorname{Im}\langle z, w \rangle - \operatorname{Re}\langle z, w \rangle \operatorname{Im}\langle z, v \rangle) + \frac{2}{\|z\|^2} \operatorname{Im}\langle w, v \rangle.$$

For  $w = Jv$ , this specializes to

$$\mathbf{d}J\mathbf{d}F(z)(v, Jv) = -\frac{2|\langle z, v \rangle|^2}{\|z\|^4} + \frac{2\|v\|^2}{\|z\|^2} = \frac{2}{\|z\|^2} \left( \|v\|^2 - \left| \left\langle \frac{z}{\|z\|}, v \right\rangle \right|^2 \right).$$

This number only depends on the length of the projection of  $v$  to the complex hyperplane orthogonal to the ray  $\mathbb{C}z$ .

Therefore there exists a smooth 2-form  $\omega$  on  $\mathbb{P}(\mathbb{C}^n)$  such that  $\pi^*\omega|_{\mathbb{S}^{2n-1}} = \mathbf{d}J\mathbf{d}F$ . It follows in particular that  $\omega$  is  $J$ -invariant, hence a  $(1, 1)$ -form, and that it is strictly positive.

Since the unitary group  $U_n(\mathbb{C})$  acts by holomorphic maps on  $\mathbb{C}^n$  and  $F$  is constant under this action, the action of  $U_n(\mathbb{C})$  preserves the Kähler structure on  $\mathbb{P}(\mathbb{C}^n)$ .

Often it is more convenient to have the form  $\omega$  in homogeneous coordinates. So let  $z \in \mathbb{C}^n$  with  $\|z\| = 1$ . We obtain a chart  $\beta: z^\perp \rightarrow \mathbb{P}(\mathbb{C}^n)$ ,  $v \mapsto [v + z]$ . We want to calculate the pullback  $\beta^*\omega$  as a  $(1, 1)$ -form on the  $(n - 1)$ -dimensional complex vector space  $z^\perp$ . Since we know already that the pullback to the unit sphere is  $\mathbf{d}J\mathbf{d}F$ , we simply have to calculate the pullback of  $\mathbf{d}J\mathbf{d}F$  under the mapping  $\beta': v \mapsto \frac{1}{\|v+z\|}(v + z)$ . Let  $x, v, w \in z^\perp$ . Then

$$\mathbf{d}\beta'(x)(v) = \frac{1}{\|x + z\|} \left( v - \frac{x + z}{\|x + z\|^2} \operatorname{Re}\langle x + z, v \rangle \right).$$

Therefore, using that the form  $\mathbf{d}J\mathbf{d}F$  depends in  $\beta'(x)$  only on the orthogonal projection of the vectors on the complex hyperplane  $\beta'(x)^\perp$  and that  $\mathbf{d}J\mathbf{d}F(\lambda z) = \frac{1}{\|\lambda\|^2} \mathbf{d}J\mathbf{d}F(z)$ , we see that

$$\begin{aligned} \mathbf{d}J\mathbf{d}F(\beta'(x))(\mathbf{d}\beta'(x)v, \mathbf{d}\beta'(x)w) &= \frac{1}{\|x + z\|^2} \mathbf{d}J\mathbf{d}F\left(\frac{x + z}{\|x + z\|}\right)(v, w) \\ &= \mathbf{d}J\mathbf{d}F(x + z)(v, w). \end{aligned}$$

Hence the pullback  $\beta^*\omega$  is simply the restriction of  $\mathbf{d}J\mathbf{d}F$  to the hyperplane  $z + z^\perp$  in  $\mathbb{C}^n$ , but this is the form  $\mathbf{d}J\mathbf{d}F_z$ , where

$$F_z(x) = F(x + z) = \frac{1}{2} \log \|x + z\|^2 = \frac{1}{2} \log(1 + \|x\|^2).$$

Here we are merely interested in linear torus actions on projective space, so we consider the following situation. Let  $V \cong \mathbb{C}^k$  be a finite dimensional complex Hilbert space,  $\mathbb{P}(V)$  the projective space endowed with the Fubini–Study metric,  $T = \mathbb{R}^n/\mathbb{Z}^n$ ,  $T_{\mathbb{C}} = (\mathbb{C}^\times)^n \cong \mathbb{C}^n/i\mathbb{Z}^n$ , and  $\pi: T_{\mathbb{C}} \rightarrow \operatorname{GL}(V)$  a holomorphic representation. Then we find an orthonormal basis  $e_1, \dots, e_k$  in  $V$  (which we use to identify  $V$  with  $\mathbb{C}^k$ ), and holomorphic characters  $\chi_1, \dots, \chi_k$  of  $(\mathbb{C}^\times)^n$  such that  $\pi(z)e_j = \chi_j(z)e_j$  for  $j = 1, \dots, k$  and  $z \in T_{\mathbb{C}}$ . We write  $\chi_j(z) = e^{-i\alpha_j(z)}$  for  $\alpha_j \in \mathfrak{t}^*$ , where  $\mathfrak{t} = \mathbf{L}(T)$ .

Let  $\Psi: \mathbb{C}^{k-1} \rightarrow \mathbb{P}(\mathbb{C}^k)$  be a coordinate chart obtained by homogeneous coordinates by  $\Psi(v') = (1 : v_2 : \dots : v_k)$ , where  $v' = (v_2, \dots, v_k)$ . Then the image of  $\Psi$  in the projective space is invariant under the induced action of the group  $T_{\mathbb{C}}$ . In homogenous coordinates the action of  $T_{\mathbb{C}}$  is given by

$$z.(1 : v_2 : \dots : v_k) = (1 : \chi_1(z)^{-1}\chi_2(z)v_2 : \dots : \chi_1(z)^{-1}\chi_k(z)v_k).$$

Let  $\pi': T_{\mathbb{C}} \rightarrow \operatorname{GL}_{k-1}(\mathbb{C})$  denote the representation defined by the characters  $\chi_2\chi_1^{-1}, \dots, \chi_k\chi_1^{-1}$ . Then the function  $F_{e_1}(v') = \frac{1}{2} \log(1 + \|v'\|^2)$  is invariant under the action of  $T$ . So we can use Proposition 3.2 to see that the action of  $T$  is Hamiltonian and with  $v = (1, v_2, \dots, v_k)$

the momentum map is given by

$$\begin{aligned}
\Phi([(1, v')])(X) &= -(J\psi(X)F)_{e_1}(v') = \mathbf{d}F_{e_1}(v')(-i\psi(X)v') = \frac{1}{1 + \|v'\|^2} \operatorname{Re}\langle v', iXv' \rangle \\
&= i \frac{\langle Xv', v' \rangle}{1 + \|v'\|^2} = \sum_{j=2}^k (\alpha_j - \alpha_1)(X) \frac{|v_j|^2}{1 + \|v'\|^2} \\
&= -\alpha_1(X) \frac{\|v'\|^2}{1 + \|v'\|^2} + \sum_{j=2}^k \alpha_j(X) \frac{|v_j|^2}{1 + \|v'\|^2} \\
&= -\alpha_1(X) + \sum_{j=1}^k \alpha_j(X) \frac{|v_j|^2}{\|v\|^2}.
\end{aligned} \tag{7}$$

Therefore

$$\Phi([(1, v')]) = \sum_{j=2}^k (\alpha_j - \alpha_1) \frac{|v_j|^2}{1 + \|v'\|^2}.$$

The next step is to investigate the image of a  $T_{\mathbb{C}}$ -orbit in projective space under the momentum map. Let  $z = x + iy + i\mathbb{Z}^n \in T_{\mathbb{C}}$ . Then

$$\Phi(z, [(1, v')]) = \frac{\sum_{j=2}^k (\alpha_j - \alpha_1) |e^{-i(\alpha_j - \alpha_1)(z)} v_j|^2}{1 + \sum_{j=2}^k |e^{-i(\alpha_j - \alpha_1)(z)} v_j|^2} = \frac{\sum_{j=2}^k (\alpha_j - \alpha_1) e^{2(\alpha_j - \alpha_1)(y)} |v_j|^2}{1 + \sum_{j=2}^k e^{2(\alpha_j - \alpha_1)(y)} |v_j|^2}.$$

Write  $\delta_\alpha$  for the Dirac measure concentrated in the point  $\alpha \in \mathfrak{t}^*$  and let

$$\mu = \delta_0 + \sum_{j=2}^k |v_j|^2 \delta_{\alpha_j - \alpha_1} = \sum_{j=1}^k |v_j|^2 \delta_{\alpha_j - \alpha_1}.$$

Then the above formula for the momentum mapping shows that

$$\Phi(z, [(1, v')]) = \mathbf{d}(\log \mathcal{L}(\mu))(2y), \tag{8}$$

where  $\mathcal{L}(\mu)$  is the Laplace transform of the measure  $\mu$  (cf. Section 1). Now the results of Section 1 make it easy for us to compute the image of the  $T_{\mathbb{C}}$ -orbit of the line  $[(1, v')]$  as the relative interior of the convex hull of the support of  $\mu$  (Corollary 1.17). Therefore

$$\Phi(T_{\mathbb{C}} \cdot [(1, v')]) = \operatorname{algint}(\operatorname{conv}\{\alpha_j : j = 1, \dots, k\}) - \alpha_1. \tag{9}$$

This result shows that the natural momentum map which was defined by the chart obtained by homogeneous coordinates depends on the choice of this chart. Of course, since  $T$  is abelian, we could easily define another momentum map by taking  $\varphi'(X) := \varphi(X) + \gamma(X)\mathbf{1}$ , where  $\mathbf{1}$  is the function constant to 1 and  $\gamma$  is a linear functional on  $\mathfrak{t}$ . Then the image of the momentum map is shifted by  $\gamma$ . In view of formula (9), it seems to be natural to take  $\gamma = \alpha_1$ . That this choice is in fact a rather natural one can be seen as follows.

We have already observed that the Fubini–Study metric on  $\mathbb{P}(\mathbb{C}^k)$  is invariant under the action of the unitary group  $U_k(\mathbb{C})$ . We will see that the action of  $U_k(\mathbb{C})$  is in fact Hamiltonian.



Writing  $\mathfrak{u}_k(\mathbb{C})$  for the Lie algebra of  $U_k(\mathbb{C})$  which consists of the space of skew-Hermitian matrices, we define for each  $X \in \mathfrak{u}_k(\mathbb{C})$  the function

$$\varphi'(X)([v]) := i \frac{\langle Xv, v \rangle}{\langle v, v \rangle}$$

on  $\mathbb{P}(\mathbb{C}^k)$  (note that the right hand side only depends on  $[v] = \mathbb{C}^\times v$ ). As we have seen above in homogeneous coordinates, this is up to a constant the correct Hamiltonian function, hence  $i\mathcal{X}_{\varphi'(X)}\omega + d\varphi'(X) = 0$ . On the other hand

$$\begin{aligned} \{\varphi'(X), \varphi'(Y)\}([v]) &= \mathcal{X}_{\varphi'(X)}\varphi'(Y)([v]) = \frac{d}{dt}\Big|_{t=0} \varphi'(Y)([e^{-tX}v]) = \frac{d}{dt}\Big|_{t=0} i \frac{\langle Ye^{-tX}v, e^{-tX}v \rangle}{\langle e^{-tX}v, e^{-tX}v \rangle} \\ &= \frac{d}{dt}\Big|_{t=0} i \frac{\langle e^{tX}Y e^{-tX}v, v \rangle}{\langle v, v \rangle} = i \frac{\langle [X, Y]v, v \rangle}{\langle v, v \rangle} = \varphi'([X, Y])([v]). \end{aligned}$$

Therefore  $\Phi'([v])(X) := \varphi'(X)([v])$  defines a momentum map for the action of  $U_k(\mathbb{C})$  on  $\mathbb{P}(\mathbb{C}^k)$ .

If  $\pi: T_{\mathbb{C}} \rightarrow GL_k(\mathbb{C})$  is a representation of the complex torus given by the characters  $\chi_1, \dots, \chi_k$  as above, we find in particular that

$$\Phi'([v])(X) = i \frac{\sum_{j=1}^k (-i\alpha_j)(X) |v_j|^2}{\langle v, v \rangle}$$

and therefore

$$\Phi'([v]) = \frac{1}{\|v\|^2} \sum_{j=1}^k |v_j|^2 \alpha_j.$$

This shows that the image of the whole projective space is the convex hull of the set  $P := \{\alpha_1, \dots, \alpha_k\}$  of all weights. But as we have already seen, much stronger results hold. To make this explicit with the new momentum map, let  $z = x + iy + i\mathbb{Z}^n \in T_{\mathbb{C}}$ . Then

$$\Phi'(z.[v]) = \frac{\sum_{j=1}^k \alpha_j e^{2\alpha_j(y)} |v_j|^2}{\sum_{j=1}^k e^{2\alpha_j(y)} |v_j|^2} = d(\log f_{\mu_v})(2y) \quad \text{for} \quad \mu_v = \sum_{j=1}^k |v_j|^2 \delta_{\alpha_j}. \quad (10)$$

**Theorem 3.4.** *Let  $T_{\mathbb{C}} = \mathbb{C}^n / i\mathbb{Z}^n$  and  $\pi$  be a holomorphic representation on  $\mathbb{C}^n$  given by the characters  $\chi_1, \dots, \chi_k$  with  $\chi_j(z) = e^{-i\alpha_j(z)}$  and  $\alpha_j \in \mathfrak{t}^*$ . Let further  $[v] \in \mathbb{P}(\mathbb{C}^k)$  and consider the holomorphic action of  $T_{\mathbb{C}}$  on  $\mathbb{P}(\mathbb{C}^k)$  induced by  $\pi$ . Then the momentum map*

$$\Phi'([v]) = \frac{1}{\|v\|^2} \sum_{j=1}^k |v_j|^2 \alpha_j$$

*for the Hamiltonian action of the torus  $T = i\mathbb{R}^n / i\mathbb{Z}^n$  on  $\mathbb{P}(\mathbb{C}^k)$  satisfies*

$$\Phi'(T_{\mathbb{C}}.[v]) = \text{algint}(P_v), \quad \text{where} \quad P_v := \text{conv}\{\alpha_j : v_j \neq 0\} \quad (11)$$

*is a polyhedron in  $\mathfrak{t}^*$ .*

*The orbit closure  $\overline{T_{\mathbb{C}}.[v]}$  is mapped onto the whole polyhedron  $P_v$ . The other  $T_{\mathbb{C}}$ -orbits are mapped onto the relative interior of the faces of  $P_v$ . This establishes a bijection between  $T_{\mathbb{C}}$ -orbits in the orbit closure and faces of  $P_v$ .*

*Proof.* The first assertion about the image of  $T_{\mathbb{C}}.[v]$  follows from (10) and Corollary 1.17. Since the orbit closure is compact, it has a compact image and since the orbit is dense (the Zariski closure and the closure in the manifold topology coincide) and  $\Phi'$  is continuous, it follows that  $\Phi'$  maps it onto  $P_v$ .

Now let  $[v']$  be contained in the closure of  $T_{\mathbb{C}}.[v]$ . Then, by taking another element in  $T_{\mathbb{C}}.[v']$  if necessary, we can assume that there exists a set  $J \subseteq \{1, \dots, k\}$  such that

$$v'_j = \begin{cases} v_j & \text{for } j \in J \\ 0 & \text{for } j \notin J \end{cases}$$

and  $F_J := \text{conv}\{\alpha_j : j \in J\}$  is a face of  $P_v$  (cf. [Ne92, Th. IV.13], [Od88, Prop. 1.6], or [Od91, p.416]). Then it follows from the first part of the theorem that

$$\Phi'(T_{\mathbb{C}}.[v']) = \text{algint}(F_J)$$

and that two different orbits are mapped into different faces.  $\square$

Orbit closures as we have seen above are called *projective toric varieties*. Therefore Theorem 3.4 establishes a connection between the projective toric variety  $\overline{T_{\mathbb{C}}.[v]}$  and the polyhedron in  $\mathfrak{t}^*$  which arises as the image of the momentum map and which describes the stratification of the orbit closure into orbits.

More information on toric varieties and momentum mappings can be found in [Ju81, At82, At83, GS82, Br85]. For a discussion of the normality of the orbits closures  $\overline{T_{\mathbb{C}}.[v]}$  we refer to [Od88, pp.95/96].

## 4 Mixed volumes and inequalities

In this section we eventually turn to the applications of the techniques explained above to the Brunn–Minkowski inequality and the Alexandrov–Fenchel inequality.

In the following we fix  $k \in \mathbb{N}$  and write  $I \subseteq \mathbb{N}_0^k$  for a multiindex  $I = (i_1, \dots, i_k)$ . We define its *degree* by  $|I| := \sum_{j=1}^k i_j \in \mathbb{N}_0$  and write  $\Delta_k^n$  for the set of all  $k$ -multiindices of degree  $n$  and  $\Delta_k^{\leq n}$  for the set of all  $k$ -multiindices with degree  $\leq n$ . We consider  $\Delta_k^{\leq n}$  as a “discrete simplex” of dimension  $k$ .

**Lemma 4.1.** *Let  $n \in \mathbb{N}$ ,  $\mathbb{K}$  a field of characteristic 0, and for a multiindex  $I$  write  $Iy := \sum_{j=1}^k i_j y_j$  for the corresponding linear form on  $\mathbb{K}^k$ . Then the polynomials  $(Iy)^n$ ,  $I \in \Delta_k^n$  form a basis of the space of homogeneous polynomials of degree  $n$  in  $y_1, \dots, y_k$ .*

*Proof.* For a multiindex  $J \in \Delta_k^n$  we write

$$b_J := \frac{n!}{j_1! \dots j_k!}$$

for the corresponding binomial coefficient. Then

$$(Iy)^n = \sum_{J \in \Delta_k^n} b_J (i_1 y_1)^{j_1} \dots (i_k y_k)^{j_k} = \sum_{J \in \Delta_k^n} b_J I^J y^J.$$

Therefore the square matrix with entries  $a_{IJ} = b_J I^J$  describes the coefficients of the polynomials  $(Iy)^n$  in the monomial basis. To show that this matrix is regular, it suffices to deal with the matrix given by  $a'_{IJ} = I^J$ . We thus have to show that  $\sum_J c_J I^J = 0$  for all  $I \in \Delta_k^n$  implies  $c_J = 0$  for all  $J$ .

This is equivalent to show that, for a homogeneous polynomial  $f(y) = \sum_{|J|=n} c_J y^J$  to vanish, it suffices to vanish on  $\Delta_k^n$ . We consider the affine map

$$\varphi: \mathbb{K}^{k-1} \rightarrow \mathbb{K}^k, \quad (y_1, \dots, y_{k-1}) \mapsto (y_1, \dots, y_{k-1}, n - y_1 - \dots - y_{k-1})$$

which maps  $\Delta_{k-1}^{\leq n}$  onto  $\Delta_k^n$ . Therefore  $f$  vanishes on  $\Delta_k^n$  if and only if the inhomogeneous polynomial  $g := f \circ \varphi$  of degree  $\leq k$  vanishes on  $\Delta_{k-1}^{\leq n}$ . We thus have to show that a polynomial  $g(y) = \sum_{J \in \Delta_{k-1}^{\leq n}} c_J y^J$  of  $k-1$  variables  $y_1, \dots, y_{k-1}$  vanishes if it vanishes on the set  $\Delta_{k-1}^{\leq n}$ . This means that the subset  $\Delta_{k-1}^{\leq n} \subseteq \mathbb{K}^{k-1}$  is a *determining subset* for the space of polynomials of degree  $\leq k-1$ .

This is done by induction. Since it is trivial for  $k=1$  or  $n=1$  (affine maps), we assume that  $k, n > 1$  and that the assertion is true for all smaller values of  $k$  and  $n$ . Suppose that  $g(I) = 0$  for all  $I \in \Delta_{k-1}^{\leq n}$ . Restricting to the hyperplanes given by  $y_\ell = 0$ , it follows from our induction hypothesis that  $c_J = 0$  if  $j_\ell = 0$  for some  $\ell$ . Therefore  $c_J \neq 0$  implies that there exists a multiindex  $J'$  with  $J = J' + (1, 1, \dots, 1)$ . Hence

$$g(y) = \sum_{J \in \Delta_{k-1}^{\leq n}} c_J y^J = y_1 \cdot \dots \cdot y_{k-1} \sum_{J' \in \Delta_{k-1}^{\leq n-k+1}} c_{J' + (1, 1, \dots, 1)} y^{J'}.$$

We conclude that the polynomial  $g'(y) = \sum_{J' \in \Delta_{k-1}^{\leq n-k+1}} c_{J' + (1, 1, \dots, 1)} y^{J'}$  vanishes on the set  $(1, \dots, 1) + \Delta_{k-1}^{\leq n-k+1}$ . By our induction hypothesis, the set  $\Delta_{k-1}^{\leq n-k+1}$  is a determining subset for the space of polynomials of degree  $\leq n-k+1$ . Hence all translates of this set are determining, and this leads to  $g' = 0$ , so that also  $c_{J' + (1, 1, \dots, 1)} = 0$  for all  $J' \in \Delta_{k-1}^{\leq n-k+1}$ , and therefore that  $c_J = 0$  for all  $J \in \Delta_{k-1}^{\leq n}$ . This completes the proof.  $\square$

The preceding lemma will permit us later to define mixed volumes via positive  $(1, 1)$ -forms. Let  $k \in \mathbb{N}$  and  $\Omega := (\omega_1, \dots, \omega_k)$  be a sequence of positive  $(1, 1)$ -forms on the compact complex manifold  $M$  of complex dimension  $n$ . For  $I \in \Delta_k^n$  we put

$$\Omega^I := \omega_1^{i_1} \wedge \dots \wedge \omega_k^{i_k}.$$

We are interested in the behaviour of the function  $I \mapsto \int_M \Omega^I$ .

**Definition 4.2.** A function  $\ell: \Delta_k^n \rightarrow \mathbb{R}$  is called *1-concave* if it is concave on every (discrete) line (isomorphic to some  $\Delta_1^{\leq m}$ ) parallel to the edges.

If, for example,  $k=2$ , then  $\Delta_2^m = \{(0, m), (1, m-1), \dots, (m, 0)\}$  and the *1-concavity* means that

$$\ell\left(\sum_j \alpha_j(j, m-j)\right) \geq \sum_j \alpha_j \ell(j, m-j)$$

whenever  $\alpha_j \geq 0$  with  $\sum_j \alpha_j = 1$  and  $\sum_j \alpha_j(j, m-j) \in \Delta_2^m$ .

**Lemma 4.3.** *To check that a function  $\ell: \Delta_k^n \rightarrow \mathbb{R}$  is concave, it suffices to check that*

$$\ell(y_0) \geq \frac{1}{2}\ell(y_-) + \frac{1}{2}\ell(y_+),$$

where  $(y_-, y_0, y_+)$  forms a part of a “discrete line segment” in  $\Delta_k^n$  which is parallel to an edge, i.e.,  $y_+ - y_0 = y_0 - y_- = e_{j_1} - e_{j_2}$  for two canonical basis vectors  $e_{j_1}, e_{j_2}$  of  $\mathbb{R}^n$ .

*Proof.* Since we only have to consider a line segment parallel to an edge of  $\Delta_k^n$ , we may w.l.o.g. assume that  $k = 2$ . Let  $I = \text{conv} \Delta_2^n \subseteq \mathbb{R}^2$ . Then we extend  $\ell$  to a continuous piecewise affine function on  $I$ . The graph of this function is a polygon and the condition imposed on  $\ell$  yields that this polygon is concave at every vertex. Therefore it is a concave polygon and this implies that  $\ell$  is a concave function.  $\square$

It is the preceding lemma which is responsible for the fact that one gets merely 1-concavity in the following theorem.

**Theorem 4.4.** (Alexandrov–Fenchel Theorem for compact complex manifolds) *Let  $M$  be a connected compact complex manifold,  $n = \dim_{\mathbb{C}} M$ , and  $\Omega = (\omega_1, \dots, \omega_k)$  a sequence of positive closed  $(1, 1)$ -forms. Then the function*

$$\ell: \Delta_k^n \rightarrow \mathbb{R}^+, \quad I \mapsto \log \int_M \Omega^I$$

*is 1-concave.*

*Proof.* For a proof we refer to [Gr90, Thm. I.6B].  $\square$

**Remark 4.5.** The preceding theorem remains true for irreducible projective varieties in the sense that one has to consider only those  $(1, 1)$ -forms  $\omega$  on the set  $M_{\text{reg}}$  of regular points which have the property that for every holomorphic map  $\beta: U \rightarrow M$ ,  $U \subseteq \mathbb{C}^n$  open, the pull-back  $\beta^*\omega$  on  $\beta^{-1}(M_{\text{reg}})$  extends smoothly to  $U$ . Again we refer to [Gr90]. The main ingredient in the proof is Hironaka’s theorem on the resolution of singularities (cf. [Hi70]).

## Applications to convex sets

**Definition 4.6.** Let  $Y_1, \dots, Y_k$  be compact convex subsets of  $\mathbb{R}^n$ . Consider Legendre functions  $f_j$  with  $\text{algint}(Y_j) = \text{df}_j(\mathbb{R}^n)$ , and write  $\omega_j := \omega_{f_j} = \text{dJdf}_j$  for the corresponding positive  $(1, 1)$ -form on the complex manifold  $M = \mathbb{C}^n / i\mathbb{Z}^n \cong T_{\mathbb{C}}$  (cf. Proposition 2.3).

For  $I \in \Delta_k^n$  the integral  $\frac{1}{n!} \int_M \Omega^I$  is called the  $I$ -th mixed volume of  $(Y_1, \dots, Y_k)$ , denoted

$$[Y^I] = [Y_1^{i_1}, \dots, Y_k^{i_k}] = \frac{1}{n!} \int_M \omega_1^{i_1} \wedge \dots \wedge \omega_k^{i_k}.$$

This definition is motivated by the following observation. For  $t_1, \dots, t_k \geq 0$  with  $\sum_j t_j > 0$ , the function  $f := t_1 f_1 + \dots + t_k f_k$  is a Legendre function with  $\text{df}(\mathbb{R}^n) = \text{algint}(Y)$

for  $Y = t_1 Y_1 + \dots + t_k Y_k$  (Lemma 1.19). By an easy induction we derive from Lemmas 1.19, 1.20 and Proposition 2.3(i) that

$$\begin{aligned}
\text{vol}(t_1 Y_1 + \dots + t_k Y_k) &= \text{vol}(\mathbf{d}(t_1 f_1 + \dots + t_k f_k)(\mathbb{R}^n)) \\
&= \int_{\mathbb{R}^n} \det(d^2(t_1 f_1 + \dots + t_k f_k))(x) dx && \text{by Lemma 1.20} \\
&= \frac{1}{n!} \int_M \omega_f^n && \text{by Proposition 2.3} \\
&= \frac{1}{n!} \int_M (t_1 \omega_1 + \dots + t_k \omega_k)^n \\
&= \frac{1}{n!} \int_M \sum_{|J|=n} b_J t^J \omega_1^{j_1} \wedge \dots \wedge \omega_k^{j_k} \\
&= \sum_{|J|=n} b_J t^J [Y_1^{j_1}, \dots, Y_k^{j_k}]
\end{aligned}$$

because  $\omega_f = \mathbf{d}J\mathbf{d}\tilde{f} = t_1 \omega_1 + \dots + t_k \omega_k$ . This calculation shows that the function

$$\mathbb{R}_+^n \rightarrow \mathbb{R}, \quad (t_1, \dots, t_n) \mapsto \text{vol}(t_1 Y_1 + \dots + t_k Y_k) = \sum_{|J|=n} b_J t^J [Y_1^{j_1}, \dots, Y_k^{j_k}] \quad (12)$$

is a polynomial of degree  $\leq n$  whose coefficients are determined by the mixed volumes. This justifies the terminology. For  $k = 1$ , we obtain in particular

$$[Y^n] = \text{vol}(Y).$$

**Lemma 4.7.** *Let  $Y_j$ ,  $f_j$ , and  $\omega_j$ ,  $j = 1, \dots, k$ , be as above. Then the following assertions hold:*

- (i) *For  $I, J \in \Delta_k^n$ , let  $c_{IJ}$  be the coefficients for which  $y^I = \sum_J c_{IJ} (Iy)^n$ . Then we have the relation*

$$\Omega^I = \sum_J c_{IJ} (j_1 \omega_1 + \dots + j_k \omega_k)^n.$$

- (ii)  $[Y^I] = \sum_J c_{IJ} \text{vol}(j_1 Y_1 + \dots + j_k Y_k)$ .

- (iii)  $\text{vol}(Y_1 + Y_2) = [(Y_1 + Y_2)^n] = \sum_{j=0}^n \binom{n}{j} [Y_1^j, Y_2^{n-j}]$ .

*Proof.* (i) First we note that the existence of the  $c_{IJ}$  follows from Lemma 4.1. Since the 2-forms  $\omega_j$  generate a commutative algebra, it follows from  $y^I = \sum_J c_{IJ} (Iy)^n$  that

$$\Omega^I = \sum_J c_{IJ} (i_1 \omega_1 + \dots + i_k \omega_k)^n.$$

- (ii) For  $f := j_1 f_1 + \dots + j_k f_k$ , we have seen above that

$$\text{vol}(j_1 Y_1 + \dots + j_k Y_k) = \frac{1}{n!} \int_M (j_1 \omega_1 + \dots + j_k \omega_k)^n.$$

Therefore (ii) follows from (i).

- (iii) is a special case of (12). □

Note that Lemma 4.7(ii) shows in particular that  $[Y^I]$  does not depend on the choice of the functions  $f_j$ .

**Theorem 4.8.** (Alexandrov–Fenchel Theorem for convex sets) *Let  $Y_1, \dots, Y_k$  be bounded convex subsets of  $\mathbb{R}^n$ . Then*

$$\ell_Y(I) := \log[Y^I]$$

*defines a 1-concave function on  $\Delta_k^n$ .*

*Proof.* Since one can approximate bounded convex sets arbitrarily well by polyhedra generated by rational points, and since  $[Y^I]$  is positively homogeneous, it suffices to prove Theorem 4.8 for polyhedra with integral extreme points. We may even assume that one extreme point of each polyhedron is the origin.

Suppose that  $Y_1 = \text{conv}\{0, \alpha_1^1, \dots, \alpha_{\ell_1}^1\}$  with  $\alpha_j^1(\mathbb{Z}^n) \subseteq 2\pi\mathbb{Z}$  for all  $j$ . Then we identify  $\mathbb{R}^n$  with  $\mathfrak{t}^*$  for  $T = i\mathbb{R}^n/i\mathbb{Z}^n$  and consider the representation of  $T_{\mathbb{C}}$  defined by the characters  $\chi_0^1(z) = 1, \chi_j^1(z) := e^{-i\alpha_j^1(z)}, j = 1, \dots, \ell_1$  on  $\mathbb{C}^{\ell_1+1}$ . Let  $v_1 = (1, \dots, 1) \in \mathbb{C}^{\ell_1+1}$  and consider the orbit closure  $M_1 := \overline{T_{\mathbb{C}} \cdot [v_1]}$  in the projective space  $\mathbb{P}(\mathbb{C}^{\ell_1+1})$ . Then the corresponding momentum map maps  $M_1$  onto  $Y_1 \subseteq \mathfrak{t}^*$  (Theorem 3.4) in such a way that the pull-back  $\tilde{\omega}_1$  of the Kähler form  $\omega_1$  on  $M_1$  via the orbit map  $T_{\mathbb{C}} \rightarrow M_1, z \mapsto z \cdot [v_1]$  can be written as  $\tilde{\omega}_1 = \mathbf{d}J\mathbf{d}F_1$ , where

$$F_1(z) = \frac{1}{2} \log(1 + \|zv_1'\|^2) = \frac{1}{2} \log\left(1 + \sum_{j=1}^{\ell_1} |\chi_j^1(z)|^2\right) = \frac{1}{2} \log\left(1 + \sum_{j=1}^{\ell_1} e^{2\alpha_j^1(y)}\right).$$

So  $F_1 = -\log \mathcal{L}(\mu)(-2y)$ , where  $\mathcal{L}(\mu)$  is the Laplace transform of the measure

$$\mu = \delta_0 + \sum_{j=1}^{\ell_1} \delta_{\alpha_j}.$$

Then  $\mathbf{d}F_1(y) = \mathbf{d}(\log \mathcal{L}(\mu))(2y)$  and therefore  $\tilde{\omega}_1$  is a positive  $(1, 1)$ -form on  $T_{\mathbb{C}}$  representing the polyhedron  $Y_1$ . We proceed similarly for  $Y_2, \dots, Y_k$ .

Now, by definition of the mixed volume,

$$[Y^I] = \frac{1}{n!} \int_{T_{\mathbb{C}}} (i_1 \tilde{\omega}_1 + \dots + i_k \tilde{\omega}_k)^n.$$

We consider the mapping

$$\beta: T_{\mathbb{C}} \rightarrow \mathbb{P}(\mathbb{C}^{\ell_1+1}) \times \dots \times \mathbb{P}(\mathbb{C}^{\ell_k+1}), \quad z \mapsto (z \cdot [v_1], \dots, z \cdot [v_k])$$

and the toric variety  $M := \overline{\beta(T_{\mathbb{C}})}$ . The projection  $\pi_j$  onto the  $j$ -th factor maps  $M$  onto  $M_j$  and the pull-back  $\pi_j^* \omega_j$  satisfies  $\tilde{\omega}_j = \beta^* \pi_j^* \omega_j$ . Therefore

$$\begin{aligned} [Y^I] &= \frac{1}{n!} \int_{T_{\mathbb{C}}} (i_1 \tilde{\omega}_1 + \dots + i_k \tilde{\omega}_k)^n = \frac{1}{n!} \int_{T_{\mathbb{C}}} \beta^* (i_1 \pi_1^* \omega_1 + \dots + i_k \pi_k^* \omega_k)^n \\ &= \frac{1}{n!} \int_M (i_1 \pi_1^* \omega_1 + \dots + i_k \pi_k^* \omega_k)^n = \frac{1}{n!} \int_M \Omega^I \end{aligned}$$

for  $\Omega = (\pi_1^* \omega_1, \dots, \pi_k^* \omega_k)$ . Now Remark 4.5 tells us that Theorem 4.4 applies and this completes the proof.  $\square$

**Lemma 4.9.** *If  $f: \Delta_k^n \rightarrow \mathbb{R}$  is a 1-concave function, then*

$$f(i_1, \dots, i_k) \geq \sum_{j=1}^k \frac{i_j}{n} f(ne_j) \quad \text{and} \quad e^{f(i_1, \dots, i_k)} \geq \prod_{j=1}^k (e^{f(ne_j)})^{\frac{i_j}{n}},$$

where  $e_1, \dots, e_k \in \Delta_k^1$  are the extreme points.

*Proof.* This is verified by induction on  $k$ . Für  $k \leq 2$  the assertion follows from the definition. So let us assume that  $k > 2$  and that it holds for  $k - 1$ . If one entry  $i_j$  vanishes, then the induction hypothesis applies directly. If this is not the case, then we observe that, for  $m := i_1 + i_2$ , the element  $(i_1, \dots, i_k)$  lies on the discrete line between

$$(m, 0, i_3, \dots, i_k) \quad \text{and} \quad (0, m, i_3, \dots, i_k).$$

Therefore 1-concavity and the induction hypothesis leads to

$$\begin{aligned} f(i_1, \dots, i_k) &\geq \frac{i_1}{m} f(m, 0, i_3, \dots, i_k) + \frac{i_2}{m} f(0, m, i_3, \dots, i_k) \\ &\geq \frac{i_1}{m} \left( \frac{m}{n} f(ne_1) + \sum_{j>2} \frac{i_j}{n} f(ne_j) \right) + \frac{i_2}{m} \left( \frac{m}{n} f(ne_2) + \sum_{j>2} \frac{i_j}{n} f(ne_j) \right) \\ &= \frac{i_1}{n} f(ne_1) + \frac{i_2}{n} f(ne_2) + \frac{i_1 + i_2}{m} \sum_{j>2} \frac{i_j}{n} f(ne_j) = \sum_{j=1}^k \frac{i_j}{n} f(ne_j). \quad \square \end{aligned}$$

With Lemma 4.9 we derive from Theorem 4.8:

**Corollary 4.10.**  $[Y^I] \geq \text{vol}(Y_1)^{\frac{i_1}{n}} \cdot \dots \cdot \text{vol}(Y_k)^{\frac{i_k}{n}}$ .

*Proof.*  $\ell_Y(I) = \ell_Y(i_1, \dots, i_k) \geq \sum_j \frac{i_k}{n} \ell_Y(0, \dots, 0, n, 0, \dots, 0)$ .  $\square$

**Corollary 4.11.** (Brunn–Minkowski inequality)

$$\text{vol}(Y_1 + Y_2)^{\frac{1}{n}} \geq \text{vol}(Y_1)^{\frac{1}{n}} + \text{vol}(Y_2)^{\frac{1}{n}}. \quad (13)$$

*Proof.* We calculate with Lemma 4.7(iii) and Corollary 4.10:

$$\begin{aligned} \text{vol}(Y_1 + Y_2) &= \sum_j \binom{n}{j} [Y_1^j, Y_2^{n-j}] \geq \sum_j \binom{n}{j} \text{vol}(Y_1)^{\frac{j}{n}} \text{vol}(Y_2)^{\frac{n-j}{n}} \\ &= \left( \text{vol}(Y_1)^{\frac{1}{n}} + \text{vol}(Y_2)^{\frac{1}{n}} \right)^n. \quad \square \end{aligned}$$

**Remark 4.12.** Let  $f_1$  and  $f_2$  be  $C^2$ -Legendre functions on  $\mathbb{R}^n$  and  $C_{f_j} = \mathbf{d}f_j(\mathbb{R}^n)$ . Then Lemma 1.20 shows that

$$\text{vol}(C_{f_j}) = \int_{\mathbb{R}^n} \det \mathbf{d}^2 f(x) \, dx.$$

Since  $C_{f_1+f_2} = C_{f_1} + C_{f_2}$  by Lemma 1.19, we see that on the level of convex functions the Brunn–Minkowski inequality reads

$$\left( \int_{\mathbb{R}^n} \det \mathbf{d}^2 (f_1 + f_2)(x) \, dx \right)^{\frac{1}{n}} \geq \left( \int_{\mathbb{R}^n} \det \mathbf{d}^2 f_1(x) \, dx \right)^{\frac{1}{n}} + \left( \int_{\mathbb{R}^n} \det \mathbf{d}^2 f_2(x) \, dx \right)^{\frac{1}{n}}. \quad (14)$$

That, conversely, this inequality also implies (13) follows from Theorem 1.18 which permits us to represent each bounded open convex set as  $C_f$  for a  $C^2$ -Legendre function  $f$  on  $\mathbb{R}^n$ .

Using Proposition 2.3, we can translate (14) into the inequality

$$\left( \int_{T_{\mathbb{C}}} (\omega_1 + \omega_2)^n \right)^{\frac{1}{n}} \geq \left( \int_{T_{\mathbb{C}}} \omega_1^n \right)^{\frac{1}{n}} \left( \int_{T_{\mathbb{C}}} \omega_2^n \right)^{\frac{1}{n}}$$

for all positive exact  $T$ -invariant  $(1,1)$ -forms on  $T_{\mathbb{C}}$ .

In the context of Kähler manifolds, one has the following version of the Brunn–Minkowski inequality.

**Theorem 4.13.** (Brunn–Minkowski inequality for Kähler manifolds) *Let  $W_1, \dots, W_k$  be compact connected  $n$ -dimensional Kähler manifolds and  $M \subseteq W := W_1 \times \dots \times W_k$  be a compact connected complex submanifold of complex dimension  $n$ ,  $\pi_j: M \rightarrow W_j$  the projections, and  $M_j := \pi_j(M)$ . Then*

$$\text{vol}(M)^{\frac{1}{n}} \geq \sum_{j=1}^k \text{vol}(M_j)^{\frac{1}{n}}. \quad (15)$$

*Proof.* Let  $\omega_j$  denote the Kähler form on  $W_j$  and  $\Omega := (\pi_1^* \omega_1, \dots, \pi_k^* \omega_k)$ . Then  $\omega := \sum_{j=1}^k \pi_j^* \omega_j$  defines the induced Kähler structure on  $M$ . Now we have

$$\begin{aligned} \text{vol}(M) &= \frac{1}{n!} \int_M \omega^n = \frac{1}{n!} \int_M \left( \sum_{j=1}^k \pi_j^* \omega_j \right)^n \\ &= \frac{1}{n!} \int_M \sum_{|I|=n} b_I \Omega^I = \frac{1}{n!} \sum_{|I|=n} b_I \int_M \Omega^I \\ &\geq \frac{1}{n!} \sum_{|I|=n} b_I \left( \int_M (\pi_1^* \omega_1)^n \right)^{\frac{i_1}{n}} \cdots \left( \int_M (\pi_k^* \omega_k)^n \right)^{\frac{i_k}{n}} \quad \text{by Corollary 4.10} \\ &= \frac{1}{n!} \left( \sum_{j=1}^k \left( \int_M (\pi_j^* \omega_j)^n \right)^{\frac{1}{n}} \right)^n = \left( \sum_{j=1}^k \left( \frac{1}{n!} \int_{M_j} \omega_j^n \right)^{\frac{1}{n}} \right)^n = \left( \sum_{j=1}^k \text{vol}(M_j)^{\frac{1}{n}} \right)^n. \quad \square \end{aligned}$$

Note that one has equality for  $n = 1$  in the preceding theorem because one trivially has equality in Theorem 4.4 in this case.

**Remark 4.14.** The preceding result remains true if we replace the Kähler manifolds  $W_j$  by projective spaces and  $M$  by an irreducible projective variety (cf. Remark 4.5 and [Gr90]).

For  $k = 2$ , we have  $M \subseteq P_1 \times P_2$  and

$$\text{vol}(M)^{\frac{1}{n}} \geq \text{vol}(M_1)^{\frac{1}{n}} + \text{vol}(M_2)^{\frac{1}{n}}.$$

As already mentioned above, we have equality for  $n = 1$ . As explained in [Gr90, §3.3], for  $n = 2$  this inequality can be derived from Hodge's inequality, resp., Hodge's index theorem in the form

$$\left( \int_M \omega_1 \wedge \omega_2 \right)^2 \geq \left( \int_M \omega_1^2 \right) \left( \int_M \omega_2^2 \right),$$



(cf. [GH78]), and for  $n \geq 3$ , this inequality is due to Hovanskii–Tessier. Therefore it is called the *Hodge–Tessier–Hovanskii inequality* (cf. [Te82, Ho84]).

For further connections of the topics of this paper with algebraic geometry, cohomology and Kähler manifolds we refer to [Gr90] and [Od88, pp.102–104].

## 5 Perspectives

In this final section we comment on some more recent developments related to the themes of [Gr90].

**Pushforwards of measure by gradients of convex functions:** One interesting issue we touched in Section 1 is writing an open convex subset  $C \subseteq \mathbb{R}^n$  as  $\mathbf{d}f(\mathbb{R}^n)$  for a  $C^2$ -Legendre function, which may be a Laplace transform  $f = \mathcal{L}(\mu)$ , where  $d\mu(y) = e^{-\|y\|^2} dy$  on  $C$  (Theorem 1.18). A slightly different issue is to consider measures on a convex set  $C$  which are the push-forward  $\mu_\psi$  under the differential  $\mathbf{d}\psi$  of a convex function  $\psi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \infty$ , of the measure  $e^{-\psi(x)} dx$ . Since  $\psi$  is locally Lipschitz, its differential  $\mathbf{d}\psi$  exists almost everywhere, so that such measures make sense for general convex functions. Here the finiteness and non-triviality of the measure  $e^{-\psi(x)} dx$  is equivalent to  $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty$ , which in turn is equivalent to the domain  $D_\psi$  having interior points and  $\lim_{x \rightarrow \infty} \psi(x) = \infty$ . In [CK15, Thm. 2] a class of convex functions is determined for which the assignment  $\psi \mapsto \mu_\psi$  leads to a bijection onto the class of finite Borel measures whose barycenter is the origin and whose support spans the whole space.

**Optimal transport:** This connects to optimal transport theory as follows. For a given measure  $\mu = \mu_\psi$ , the differential  $\mathbf{d}\psi$  is the *quadratic optimal map*, or *Brenier map*, between the measure  $e^{-\psi(x)} dx$  on  $\mathbb{R}^n$  and the measure  $\mu$  ([Br91]). This refers to the existence of the unique *polar factorization* of a measurable map  $u: X \rightarrow \mathbb{R}^n$  from a probability space  $(X, \mu)$  to  $\mathbb{R}^n$  in the form  $u(x) = \mathbf{d}\psi(s(x))$ , where  $\Omega \subseteq \mathbb{R}^n$  is a bounded domain endowed with the normalized Lebesgue measure  $\mu_\Omega$ ,  $s: (X, \mu) \rightarrow (\Omega, \mu_\Omega)$  is measure preserving, and  $\psi: \Omega \rightarrow \mathbb{R}$  is convex ([Br91]). Note that, for  $\psi(x) = \frac{1}{2}\|x\|^2$ , the measure  $e^{-\psi(x)} dx$  simply is the Gaussian measure on  $\mathbb{R}^n$ . For applications of the measures  $\mu_\psi$  and momentum maps to Poincaré type inequalities in analysis, we refer to [Kl13].

**Generalizations and applications of the Brunn–Minkowski inequality (BMI):** The BMI for bounded subsets  $Y_0, Y_1 \subseteq \mathbb{R}^n$  implies with  $a^2 + b^2 \geq 2ab$  for  $a, b \geq 0$  the inequality

$$\text{vol}\left(\frac{Y_0 + Y_1}{2}\right) \geq \text{vol}(Y_0)^{\frac{1}{2}} \text{vol}(Y_1)^{\frac{1}{2}}.$$

This in turn implies that, for  $Y_t := (1-t)Y_0 + tY_1$ , the function  $\log(\text{vol}(Y_t))$  is concave. This observation can be generalized as follows. In  $\mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$ , we consider the convex subset

$$Y := \bigcup_{0 \leq t \leq 1} \{t\} \times Y_t.$$

Then the function  $t \mapsto \text{vol}(Y_t)$  is a marginal of Lebesgue measure restricted to  $Y$ . The log-concavity of this function is a special case of *Prékopa’s Theorem* asserting that, for any

convex function  $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R} \cup \{\infty\}$  with  $\int_{\mathbb{R}^{n+1}} e^{-F} < \infty$ , the function

$$t \mapsto \log \int_{\mathbb{R}^n} e^{-F(t, \mathbf{x})} d\mathbf{x}$$

is concave. In fact, if  $F_Y$  is the convex indicator function of  $Y$  which is 0 on  $Y$  and  $\infty$  elsewhere, then the right hand side specializes to  $\log(\text{vol}(Y_t))$  (see [CK12]).

For the connections of the BMI and its generalizations in various branches of mathematics, we refer to Gardner's nice survey [Ga02]. Here one finds in particular a discussion of the BMI for non-convex subsets, the situations where equality holds, and its applications to isoperimetric inequalities and estimates in analysis. Moreover, generalizations of the BMI to the sphere, hyperbolic space, Minkowski space and Gauss space (euclidean space where the volume is measured with respect to a Gaussian density) are explained. For subsets  $A, B$  of the integral lattices  $\mathbb{Z}^n$ , analogs of the BMI giving lower bounds of the cardinality of  $|A + B|$  in terms of  $|A|$  and  $|B|$  can be found in [GG01].

Connections to the representation theory of reductive groups have been established by the work of V. Okounkov who used the BMI to study weight polytopes ([Ok96]).

**Infinite dimensional convex geometry:** There exist natural infinite dimensional contexts in which substantial portions of the duality theory for convex functions work. Here one may start with a real bilinear duality pairing  $(\cdot, \cdot): V \times W \rightarrow \mathbb{R}$  of two infinite dimensional real vector spaces  $V$  and  $W$ . This pairing defines natural (weak) locally convex topologies on  $V$  and  $W$ . Accordingly, one may consider convex functions  $f: V \rightarrow \mathbb{R} \cup \{\infty\}$ , define closedness in terms of the weak closedness of the epigraph  $\text{epi}(f) \subseteq V \times \mathbb{R}$  and consider the conjugate convex function  $f^*: W \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $f^*(w) := \sup_{v \in V} (v, w) - f(v)$  which is automatically closed. However, for more refined applications, it is important to also have a finer locally convex topology on  $V$  for which the domain of  $f$  has interior points. For more details, further developments and applications see [Mi08], [Bou07] and [Ro70, §3].

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